Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

A Discrete Vortex Simulation of Kelvin-Helmholtz Instability

I. G. Bromilow* and R. R. Clements†
University of Bristol, Bristol, England

I. Introduction

At the interface between two flows of different density and velocity, there is a generation of vorticity which must be included when calculating the development of the interface. In the past, models have been presented which include a time variation of vorticity due to buoyancy and/or surface tension effects, but the resulting numerical schemes were either complicated[1] or else limited the development of the vortex sheet to early times.[2] Other methods were due to Daly[3] and Baker et al.[4]

Zalosh[2] applied the discrete vortex method to the study of Kelvin-Helmholtz instability. This technique approximates a continuous distribution of vorticity by a set of discrete vortices, whose subsequent motion enables the motion of the vortex sheet to be inferred.[5] The singularity in the velocity field of a point vortex is associated with a growing randomness of the discrete vortices with their close approach. According to Zalosh, this was responsible for the inaccuracies in his calculations.

Bromilow and Clements[6] presented techniques for reducing this irregular movement. An amalgamation technique, to replace clusters of vortices by one vortex at the vorticity centroid, and a redistribution technique, to replace an unevenly distributed set of vortices by an equivalent but evenly distributed set, thus mitigating their close approach. The latter technique ensures that the circulation associated with a given segment is conserved under stretching of the sheet. The stabilizing effects of surface tension and buoyancy delay the rolling up of the sheet and so amalgamation is not incorporated here. The use of a double parametric cubic spline in the redistribution technique enables sheet curvature and gradients to be determined (correct to third order) from the spline coefficients.

II. Mathematical Model and Numerical Scheme

Consider the vortex sheet separating two incompressible flows with velocities U_1, U_2 and densities ρ_1, ρ_2 ($\rho_1<\rho_2$), respectively.

Corresponding to the normal mode in the solution of a first-order perturbation analysis,[7] the vortex sheet is given an initial sinusoidal perturbation of amplitude α. One wavelength of the vortex sheet is divided into n segments of length Δs_i with the total circulation of a segment represented by a vortex of strength γ_i placed at its midpoint (ξ_i,η_i). The inclination of the ith segment is θ_i.

Received June 22, 1982; revision received Dec. 16, 1982.

*Research Student; currently, Mathematician, Koninklijke/Shell Laboratorium, Amsterdam, the Netherlands.

†Lecturer, Department of Engineering Mathematics.

For the preceding arrangement of vortices, expressions for the components of induced velocity at the position of the ith vortex can be determined[7] and the vortex strengths k_i are obtained from the result of the perturbation analysis performed by Hama and Burke.[8]

Zalosh derived the following equation for the rate of change of γ_i with time, in terms of Froude number F_r and Weber number W_e.

$$\frac{d\gamma_i}{dt}=F_r^{-2}\Delta s_i\left(\sin\theta_i+\frac{dV_{is}}{dt}\right)-W_e^{-1}\Delta s_i\frac{\partial R_i^{-1}}{\partial s} \tag{1}$$

where

$$F_r^{-2}=\frac{g\lambda(\rho_2-\rho_1)}{(\Delta U)^2(\rho_1+\rho_2)},\quad W_e^{-1}=\frac{\sigma}{(\rho_1+\rho_2)\lambda(\Delta U)^2}$$

V_{is} is the tangential velocity component of the ith vortex, $\Delta U=U_1-U_2$,

$$R_i^{-1}=\frac{\partial^2\eta}{\partial\xi^2}\Bigg/\left[1+\left(\frac{\partial\eta}{\partial\xi}\right)^2\right]^{3/2}\Bigg|_{\xi_i} \tag{2}$$

is the curvature of the sheet, σ the coefficient of surface tension, and g the acceleration due to gravity.

In order to follow the development of the vortex sheet, three coupled ordinary-differential equations, i.e., Eq. (1) and the equations for the components of induced velocity, must be solved.

Zalosh evaluated θ_i, Δs_i, R_i^{-1} by finite difference techniques. The use of the double parametric cubic spline in the redistribution of vorticity enables these terms to be calculated from the spline coefficients, correct to third order.

After redistribution, the components of induced velocity $\{(u_i,v_i)\}$ (Rosenhead[7]) are evaluated for the set of vortices of strength $\{\gamma_i\}$ and positions $\{(\xi_i,\eta_i)\}$. The tangential velocity component V_{is} for each vortex is calculated from

$$V_{is}=u_i\cos\theta_i+v_i\sin\theta_i$$

Equations for the components of induced velocity are then integrated using a fourth-order Runge-Kutta integration scheme to provide the new vortex positions $\{z_i^*\}=\{(\xi_i^*,\eta_i^*)\}$ (the superscript * denotes variables evaluated at the new time step). The current values of induced velocity $\{(u_i^*,v_i^*)\}$ are also evaluated.

A double parametric cubic spline through the vortex positions provides the values of θ_i^*, $d\eta/d\xi\,|_{z_i^*}$, and $d^2\eta/d\xi^2\,|_{z_i^*}$. The new tangential velocity components $\{V_{is}^*\}$ are evaluated and, hence, dV_{is}^*/dt is evaluated by a simple backward difference. The curvatures $\{R_i^{-1*}\}$ follow from Eq. (2) and the distance s_i^* of each vortex, measured along the sheet from the start of the wavelength z_0, can also be determined. $\partial R^{-1}/\partial s|_{z_i^*}$ is determined from $\{s_i^*\}$ and $\{R_i^{-1*}\}$ by fourth-order Lagrangian interpolation. The new vortex strengths $\{\gamma_i^*\}$ follow after integration of Eq. (1).

III. Kelvin-Helmholtz Instability

The initial disturbance of interest is that which maximizes the growth rate of the interface. According to linear stability theory,[9] the corresponding wavenumber is $k=\sqrt{g(\rho_2-\rho_1)/\sigma}$. Incorporating this into the definitions of F_r, W_e leads to $W_e^{-1}=F_r^2/4\pi^2$. This relation is used in the following. Two cases have been chosen for presentation; these correspond to stable and marginally unstable conditions.

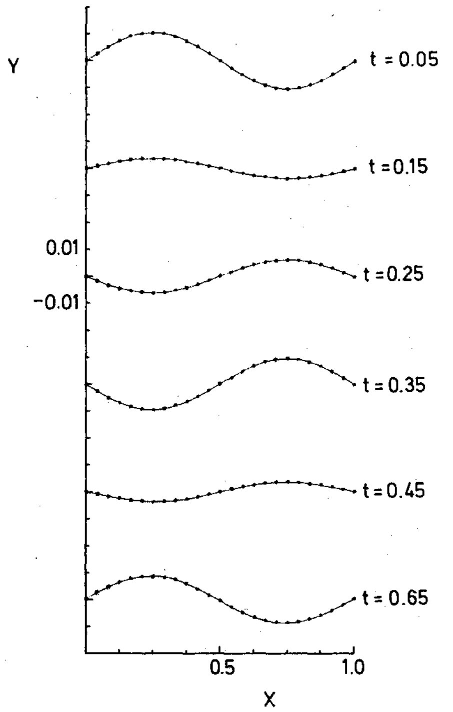

Fig. 1 Development of interface, stable conditions; $Fr^{-2}=10$, $S=0.9$, and $\alpha=0.01$.

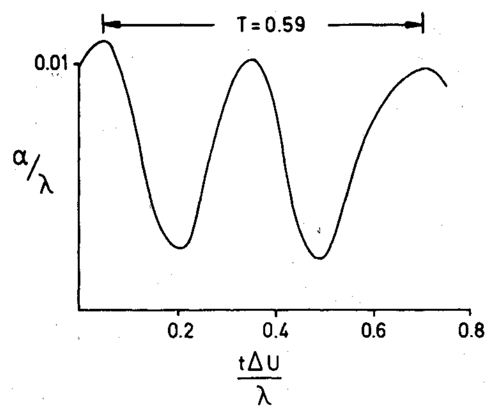

Fig. 2 Amplitude variation with time, stable conditions; $Fr^{-2}=10$, $S=0.9$, and $\alpha=0.01$.

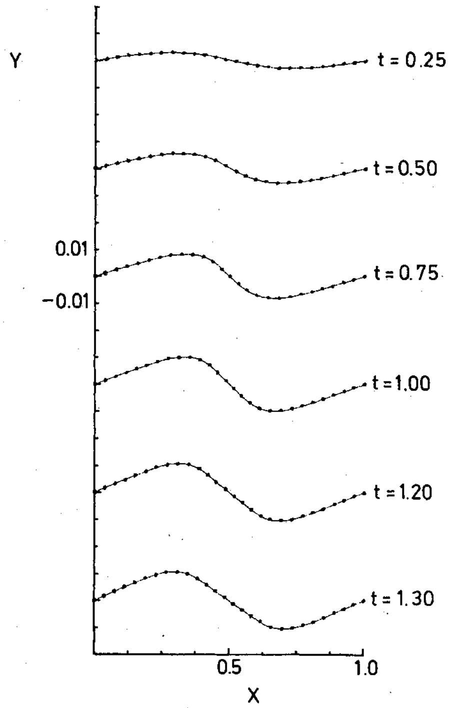

Fig. 3 Development of interface, marginally unstable conditions; $Fr^{-2}=0.7$, $S=0.99$, and $\alpha=0.01$.

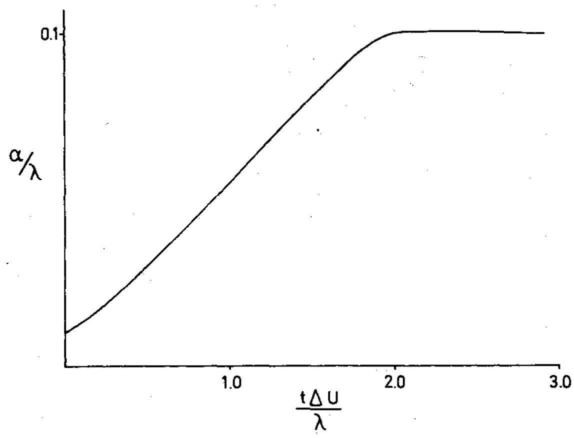

Fig. 4 Amplitude variation with time, marginally unstable conditions; $Fr^{-2}=0.7$, $S=0.99$, and $\alpha=0.01$.

Figure 1 shows the development of the interface, at different times, for $F_r^{-2}=10(W_e^{-1}=0.253)$, $\rho_1/\rho_2=0.9$, $\alpha=0.01$, and, as predicted by Drazin,[10] a standing wave develops. Linear stability theory[9] predicts that the period of oscillation will be 0.59. Figure 2 shows the amplitude variation with time from which the period is determined to be approximately 0.59.

Drazin[10] considered the development of the interface in the limit as $\rho_1 \rightarrow \rho_2$. An infinitesimal perturbation is predicted to oscillate with a long period and a maximum amplitude somewhat less than $16[\{F_r\pi^{1/2}-2\}/10\pi^2]^{1/2}$, for the present notation. Thus, taking $\rho_1/\rho_2=0.99$, $F_r^2=1.438$, $\alpha=0.01$, the maximum amplitude should be less than 0.14. In Fig. 3, the development of the interface is presented and in Fig. 4, the corresponding amplitude variation with time is shown. A maximum of 0.106 was found to exist.

According to Zalosh, a test of the accuracy of the calculation would be the change in the total circulation per wavelength since, from Eq. (1) this should be zero. Zalosh terminated his calculation when this sum differed from the initial value of 5%. For the present calculation, even after 120 steps, the total circulation is within 0.5% of the initial value.

IV. Conclusions

The use of rediscretization enables the calculation to continue beyond the times of previous calculations with a high degree of accuracy. On comparison with the results of Zalosh, the irregular movement of the vortices, the time limit of the calculation, and associated inaccuracies in predictions were not evident here. The results exhibited both quantitative and qualitative agreement with analytical predictions. These overall improvements are a consequence of rediscretization which prevents the close approach of vortices and also enables the implementation of higher order numerical schemes through the incorporation of cubic splines.

Due to the delay in rolling up the sheet, the errors in the calculations of Zalosh were probably due to the use of a sparse finite difference grid and lower order numerical schemes to calculate terms appearing in Eq. (1).

Finally, the predictions of Chandrasekhar[9] and Drazin,[10] cited in this Note, resulted from a linear stability analysis. The discrete vortex simulations have demonstrated the validity of the predictions to a nonlinear calculation.

Acknowledgments

One of the authors (I. G. Bromilow) was supported by a Science and Engineering Research Council CASE award in conjunction with Rolls-Royce Ltd., Bristol, U.K. The help from both organizations is gratefully acknowledged.

References

[1]Zaroodny, S. J. and Greenberg, M. D., "On a Vortex Sheet Approach to the Numerical Calculation of Water Waves," *Journal of Computational Physics*, Vol. 11, March 1973, pp. 440-446.

[2]Zalosh, R. C., "Discretized Simulation of Vortex Sheet Evolution with Buoyancy and Surface Tension Effects," *AIAA Journal*, Vol. 14, Nov. 1976, pp. 1517-1523.

[3]Daly, B. J., "A Technique for Including Surface Tension Effects in Hydrodynamic Calculations," *Journal of Computational Physics*, Vol. 4, 1969, pp. 97-117.

[4]Baker, G. R., Meiron, D. I., and Orszag, S. A., "Vortex Simulations of the Rayleigh-Taylor Instability," *Physics of Fluids*, Vol. 23, Aug. 1980, pp. 1485-1490.

[5]Clements, R. R. and Maull, D. J., "The Representation of Sheets of Vorticity by Discrete Vortices," *Progress in Aerospace Sciences*, Vol. 16, 1975, pp. 129-146.

[6]Bromilow, I. G. and Clements, R. R., "Some Techniques for Extending the Application of the Discrete Vortex Method of Flow Simulation," *Aeronautical Quarterly*, Vol. 33, May 1982, pp. 73-89.

[7]Rosenhead, L., "The Formation of Vortices from a Surface of Discontinuity," *Proceedings of the Royal Society*, Ser. A, Vol. 134, 1931, pp. 170-192.

[8]Hama, F. R. and Burke, E. R., "On the Rolling up of a Vortex Sheet," University of Maryland Technical Note BN-220, 1960.

[9]Chandrasekhar, S., *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press, 1961, pp. 428-435.

[10]Drazin, P. G., "Kelvin-Helmholtz Instability of Finite Amplitude," *Journal of Fluid Mechanics*, Vol. 42, June 1970, pp. 321-335.

The Two-Dimensional Laminar Wake with Initial Asymmetry

Anthony Demetriades*
Montana State University, Bozeman, Montana

A FREQUENT problem in laser cavity flows, airfoil aerodynamics, and similar two-dimensional fluid problems is the mixing of two identical uniform, compressible streams past a sharp trailing edge (TE) of a partition, on one side of which the laminar boundary layer is thicker than it is on the other. Since the streams are identical, this configuration resembles an ordinary two-dimensional wake; however, the disparity in the boundary-layer thicknesses is bound to introduce asymmetries in the flow, especially beyond but near the trailing edge. The distance beyond the latter over which such asymmetries persist is of significance, especially if no restrictions are placed on the flow Mach number M_e and the temperature ratio T_w/T_{oe} of the solid partition relative to the stagnation temperature of the stream. This Note aims at presenting formulas valid for such an asymmetric wake for any M_e and T_w/T_{oe} and any degree of asymmetry introduced via the ratio $P=\theta_1/\theta_2$, where θ is the boundary-layer momentum thickness at the trailing edge and subscripts *1* and *2* refer to the two sides of the partition.

Received Feb. 12, 1982; revision received Dec. 20, 1982.

*Professor, Department of Mechanical Engineering. Associate Fellow AIAA.

For initially asymmetric wakes (arbitrary $0<P<\infty$) the method used here is based on the linearized analysis of Gold[1] for wakes with arbitrary initial profiles. Gold's analysis used the Oseen linearization and a Chapman-Rubesin factor of unity. The Prandtl number, left arbitrary by Gold, was here assumed to be unity, to be consistent with the use of the Crocco relation connecting the fluid, wall (or TE), and stream (or wake "edge") temperatures T, T_w, and T_e, respectively,

$$\frac{T}{T_e}=\frac{T_w}{T_e}+\left(1+\frac{\gamma-1}{2}M_e^2-\frac{T_w}{T_e}\right)\frac{u}{u_e}-\frac{\gamma-1}{2}M_e^2\left(\frac{u}{u_e}\right)^2 \quad (1)$$

where u is the flow velocity and $(\ )_e$ denotes the edge conditions. The specific heat ratio γ was taken to be 1.4 in the computations shown below and the pressure was assumed to be everywhere constant.

The present calculation will be shown here in summary, since its details can be found in Ref. 2. The stretched longitudinal and lateral distances are given in terms of the physical coordinates x^* and y^* and the Reynolds number

$$Re_\Theta\equiv\frac{u_e(\theta_1+\theta_2)}{\nu_e}=\frac{u_e}{\nu_e}\Theta,\quad(\Theta\equiv\theta_1+\theta_2) \quad (2)$$

by

$$x\equiv\frac{x^*}{\Theta Re_\Theta},\quad y\equiv\frac{1}{\Theta}\int_0^{y^*}\frac{\rho^*}{\rho_e^*}\mathrm{d}y^*\equiv\frac{\tilde{y}}{\Theta} \quad (3)$$

where ρ^* is the density. If the velocity profiles assumed at the TE ($x=x^*=0$) are of the type

$$y^*,y>0:\quad 1-\frac{u}{u_e}=e^{-\tilde{y}/\theta_1}=e^{-\frac{P+1}{P}y} \quad (4)$$

$$y^*,y<0:\quad 1-\frac{u}{u_e}=e^{\tilde{y}/\theta_2}=e^{(P+1)y} \quad (5)$$

then one can obtain

$$\begin{aligned}\bar{u}\equiv 1-\frac{u}{u_e}&=\bar{u}(x,y;\ M_e,T_w/T_{oe},P)\\&=\frac{1}{2}\Big\{\exp(P+1)^2x+(P+1)y\\&\quad\times\left[1-\mathrm{Erf}\left((P+1)\sqrt{x}+\frac{y}{2\sqrt{x}}\right)\right]\\&\quad+\left(\exp\left[\left(\frac{P+1}{P}\right)^2x-\left(\frac{P+1}{P}\right)y\right]\right)\\&\quad\times\left[1+\mathrm{Erf}\left(\frac{y}{2\sqrt{x}}-\frac{P+1}{P}\sqrt{x}\right)\right]\Big\}\end{aligned} \quad (6)$$

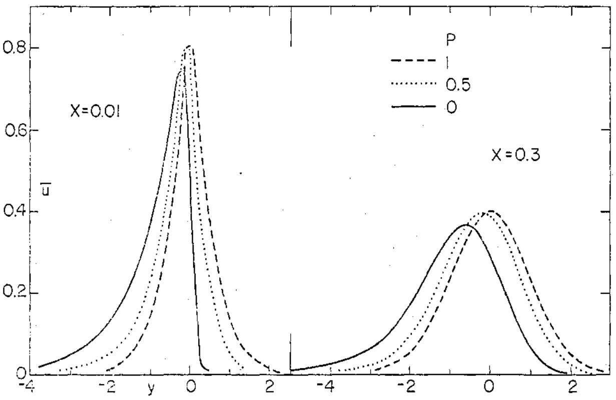

Fig. 1 Wake profiles as a function of $P=\theta_1/\theta_2$ and distance x from the trailing edge.